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LECTURES ON CHERN-WEIL THEORY AND WITTEN DEFORMATIONS

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Chapter 5

Morse Inequalities: an Analytic Proof

In this chapter, we present Witten's analytic proof of Morse inequalities by refining some of the arguments in Chapter 4.

Witten's original paper [W1] has been very influential in various aspects in topology, geometry and mathematical physics. We will mention some of them in Section 5.7.

We recommend the book of Milnor [Mi] for a beautiful account of some of the classical aspects of Morse theory.

As in Chapter 4, we will work with real coefficients in this chapter.

5.1 Review of Morse Inequalities

Let M be an n-dimensional closed oriented manifold. Let $f \in C^\infty(M)$ be a smooth function on M. A point $x \in M$ is called a **critical point** of f if

$$df(x) = 0.$$

If $x \in M$ is a critical point of f, then we say x is **nondegenerate** if the Hessian of f at x is non-singular, i.e.,

$$\det(\mathrm{Hess}_f(x)) \neq 0.$$

It is easy to verify that every nondegenerate critical point $x \in M$ of f is isolated, that is, there is no other critical point of f in a sufficiently small open neighborhood of $x \in M$.

A smooth function on M is called a **Morse function** if all the critical points of this function are nondegenerate. It is well-known (cf. [Mi]) that there always exists a Morse function on M. Clearly, a Morse function on a closed manifold has only a finite number of critical points.

From now on we assume f is a Morse function on M.

The following Morse lemma (cf. [Mi]) is important in many aspects of the theory of Morse functions.

Lemma 5.1 *For any critical point $x \in M$ of the Morse function f, there is an open neighborhood U_x of x and an oriented coordinate system $\mathbf{y} = (y^1, \ldots, y^n)$ such that on U_x, one has*

$$f(\mathbf{y}) = f(x) - \frac{1}{2}\left(y^1\right)^2 - \cdots - \frac{1}{2}\left(y^{n_f(x)}\right)^2 + \frac{1}{2}\left(y^{n_f(x)+1}\right)^2 + \cdots + \frac{1}{2}\left(y^n\right)^2. \tag{5.1}$$

We call the integer $n_f(x)$ the **Morse index** of f at x. Also, for later use, we assume that for any two different critical points x, $y \in M$ of f, $U_x \cap U_y = \emptyset$.

Now for any integer i such that $0 \leq i \leq n$, let β_i denote the i-th Betti number $\dim H^i_{\mathrm{dR}}(M; \mathbf{R})$. Let m_i denote the number of critical points $x \in M$ of f such that $n_f(x) = i$.

The **Morse inequalities**, for which an analytic proof will be given in this chapter, can be stated as follows.

Theorem 5.2 *(i)* **Weak Morse inequalities**: *For any integer i such that $0 \leq i \leq n$, one has*

$$\beta_i \leq m_i. \tag{5.2}$$

(ii) **Strong Morse inequalities**: *For any integer i such that $0 \leq i \leq n$, one has*

$$\beta_i - \beta_{i-1} + \cdots + (-1)^i \beta_0 \leq m_i - m_{i-1} + \cdots + (-1)^i m_0. \tag{5.3}$$

Moreover,

$$\beta_n - \beta_{n-1} + \cdots + (-1)^n \beta_0 = m_n - m_{n-1} + \cdots + (-1)^n m_0. \tag{5.4}$$

Clearly, (5.2) is a consequence of (5.3).*

*In fact, one can apply (5.3) twice to i and $i-1$ respectively, and then take sum to get (5.2).

We refer to the book [Mi] for a topological proof of this result. In the rest of this chapter, we will present an analytic proof of it by following an idea of Witten [W1].

5.2 Witten Deformation

Recall from Section 1.1 the definition of the de Rham complex

$$(\Omega^*(M), d) : 0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\dim M}(M) \longrightarrow 0.$$

Given the Morse function f, inspired by considerations in physics, Witten [W1] suggested to deform the exterior differential operator d as follows: for any $T \in \mathbf{R}$, set

$$d_{Tf} = e^{-Tf} d e^{Tf}. \tag{5.5}$$

Since $d^2 = 0$, from (5.5) one has

$$d_{Tf}^2 = 0. \tag{5.6}$$

Thus, one can deform the de Rham complex $(\Omega^*(M), d)$ to the complex $(\Omega^*(M), d_{Tf})$ defined by

$$(\Omega^*(M), d_{Tf}) : 0 \longrightarrow \Omega^0(M) \xrightarrow{d_{Tf}} \Omega^1(M) \xrightarrow{d_{Tf}} \cdots \xrightarrow{d_{Tf}} \Omega^{\dim M}(M) \longrightarrow 0.$$

Let

$$H^*_{Tf,\mathrm{dR}}(M; \mathbf{R}) = \frac{\ker d_{Tf}}{\operatorname{Im} d_{Tf}}$$

be the corresponding cohomology, with the $\mathbf{Z}$-grading given by

$$H^*_{Tf,\mathrm{dR}}(M; \mathbf{R}) = \bigoplus_{i=0}^{n} H^i_{Tf,\mathrm{dR}}(M; \mathbf{R}),$$

where for each integer i such that $0 \leq i \leq n$,

$$H^i_{Tf,\mathrm{dR}}(M; \mathbf{R}) = \frac{\ker d_{Tf}|_{\Omega^i(M)}}{\operatorname{Im} d_{Tf}|_{\Omega^{i-1}(M)}}.$$

The first important result for the Witten deformation is as follows.

Proposition 5.3 *For any integer i such that $0 \le i \le n$,*

$$\dim H^i_{Tf,\mathrm{dR}}(M;\mathbf{R}) = \dim H^i_{\mathrm{dR}}(M;\mathbf{R}).$$

Proof. For any $\alpha \in \Omega^i(M)$ such that $d\alpha = 0$, one verifies that

$$d_{Tf}(e^{-Tf}\alpha) = 0,$$

while for any $\beta \in \Omega^{i-1}(M)$, one has

$$e^{-Tf} d\beta = d_{Tf}(e^{-Tf}\beta).$$

Thus, the map

$$\alpha \in \Omega^i(M) \mapsto e^{-Tf}\alpha \in \Omega^i(M)$$

induces a well-defined homomorphism from $H^i_{\mathrm{dR}}(M;\mathbf{R})$ to $H^i_{Tf,\mathrm{dR}}(M;\mathbf{R})$.

Similarly, one sees easily that the map

$$\alpha \in \Omega^i(M) \mapsto e^{Tf}\alpha \in \Omega^i(M)$$

induces a well-defined homomorphism from $H^i_{\mathrm{dR}}(M;\mathbf{R})$ to $H^i_{Tf,\mathrm{dR}}(M;\mathbf{R})$.

It is now easy to verify that these two induced homomorphisms on cohomologies are in fact isomorphisms each of which is the inverse of the other one. □

5.3 Hodge Theorem for $(\Omega^*(M), d_{Tf})$

Let g^{TM} be a metric on TM. Recall that the Hodge theorem for the de Rham complex $(\Omega^*(M), d)$ has been reviewed in Section 4.1.

Since $T \in \mathbf{R}$, by (4.4) one deduces that for any α, $\beta \in \Omega^*(M)$,

$$\langle d_{Tf}\alpha, \beta\rangle = \left\langle e^{-Tf} d e^{Tf}\alpha, \beta\right\rangle = \left\langle \alpha, e^{Tf} d^* e^{-Tf}\beta\right\rangle.$$

Thus,

$$d^*_{Tf} := e^{Tf} d^* e^{-Tf} \tag{5.7}$$

is the formal adjoint of d_{Tf}.

Recall that $D = d + d^*$. For any $T \ge 0$, set

$$D_{Tf} = d_{Tf} + d^*_{Tf}, \tag{5.8}$$

$$\Box_{Tf} = D^2_{Tf} = d_{Tf} d^*_{Tf} + d^*_{Tf} d_{Tf}. \tag{5.9}$$

By (5.5) and (5.7), one sees that $\square_{Tf}$ preserves each $\Omega^i(M)$, $0 \leq i \leq n$. Moreover, one can well establish the Hodge theorem for the complex $(\Omega^*(M), d_{Tf})$, a consequence of which implies that for any integer i such that $0 \leq i \leq n$,

$$\dim\left(\ker \square_{Tf}|_{\Omega^i(M)}\right) = \dim H^i_{Tf,\mathrm{dR}}(M;\mathbf{R}) = \dim H^i_{\mathrm{dR}}(M;\mathbf{R}), \quad (5.10)$$

where the last equality follows from Proposition 5.3.

From (5.10), one sees that to obtain the information about the β_i's, one may take $T \to +\infty$ and study the behaviour of $\square_{Tf}$ under the limit.

5.4 Behaviour of $\square_{Tf}$ Near the Critical Points of f

Without loss of generality, we assume that on the open neighborhood U_x of a critical point $x \in M$ of f, with the coordinate system $\mathbf{y} = (y^1, \ldots, y^n)$ which are defined in Section 5.1, one has

$$g^{TM} = \left(dy^1\right)^2 + \cdots + (dy^n)^2 . \quad (5.11)$$

From (4.14)-(4.16), (5.5), (5.7) and (5.8), one verifies directly that

$$d_{Tf} = d + T df \wedge, \quad d^*_{Tf} = d^* + T i_{df}$$

and

$$D_{Tf} = D + T\widehat{c}(df), \quad (5.12)$$

where we identify df with its corresponding element in $\Gamma(TM)$ determined by g^{TM}.

Clearly, (5.12) is a special case of the deformation (4.17) in Section 4.3. However, the deformation operator in (5.12) has the advantage that the square of it preserves the $\mathbf{Z}$-grading of $\Omega^*(M)$, while the square of the deformation operator in (4.17) only preserves the $\mathbf{Z}_2$-grading of $\Omega^*(M)$, in general.

Now by the Morse lemma 5.1, one verifies that on each U_x, one has

$$df(x) = -y^1 dy^1 - \cdots - y^{n_f(x)} dy^{n_f(x)} + y^{n_f(x)+1} dy^{n_f(x)+1} + \cdots + y^n dy^n. \quad (5.13)$$

Let $e_i = \frac{\partial}{\partial y^i}$, $1 \leq i \leq n$, be the oriented orthonormal basis of TU_x.

By (5.11)-(5.13) and the Bochner type formula (4.19), one deduces that on each U_x,

$$\square_{Tf} = -\sum_{i=1}^{n}\left(\frac{\partial}{\partial y^i}\right)^2 - nT + T^2|\mathrm{y}|^2$$

$$+T\sum_{i=1}^{n_f(x)}\left(1 - c(e_i)\widehat{c}(e_i)\right) + T\sum_{i=n_f(x)+1}^{n}\left(1 + c(e_i)\widehat{c}(e_i)\right)$$

$$= -\sum_{i=1}^{n}\left(\frac{\partial}{\partial y^i}\right)^2 - nT + T^2|\mathrm{y}|^2$$

$$+2T\left(\sum_{i=1}^{n_f(x)} i_{e_i} e_i^* \wedge + \sum_{i=n_f(x)+1}^{n} e_i^* \wedge i_{e_i}\right). \tag{5.14}$$

It is easy to verify that the linear operator

$$\sum_{i=1}^{n_f(x)} i_{e_i} e_i^* \wedge + \sum_{i=n_f(x)+1}^{n} e_i^* \wedge i_{e_i}$$

is nonnegative, with the kernel being one dimensional and generated by

$$dy^1 \wedge \cdots \wedge dy^{n_f(x)}.$$

One then gets the following **Z**-graded refinement of Proposition 4.9 in the current situation.

Proposition 5.4 *For any $T > 0$, the operator*

$$-\sum_{i=1}^{n}\left(\frac{\partial}{\partial y^i}\right)^2 - nT + T^2|\mathrm{y}|^2 + 2T\left(\sum_{i=1}^{n_f(x)} i_{e_i} e_i^* \wedge + \sum_{i=n_f(x)+1}^{n} e_i^* \wedge i_{e_i}\right)$$

acting on $\Gamma(\Lambda^*(\mathrm{E}_n^*))$ is nonnegative. Its kernel is of dimension one and is generated by

$$\exp\left(\frac{-T|\mathrm{y}|^2}{2}\right) \cdot dy^1 \wedge \cdots \wedge dy^{n_f(x)}.$$

Moreover, all the nonzero eigenvalues of this operator are greater than CT for some fixed constant $C > 0$.

5.5 Proof of Morse Inequalities

Recall that in the proof of the Poincaré-Hopf index formula in Section 4.6, we have used the deformation (4.38) to reduce the proof to a finite dimensional situation. However, if we would apply this deformation to the operator D_{Tf} now, we would see that the Laplacians of the deformed operators only preserve the $\mathbf{Z}_2$-grading of $\Omega^*(M)$, not the required $\mathbf{Z}$-grading nature. Thus, one should deal with more refined arguments.

Following Witten [W1], we will instead prove the following result, from which the Morse inequalities will follow.

Proposition 5.5 *For any $c > 0$, there exists $T_0 > 0$ such that when $T \geq T_0$, the number of eigenvalues in $[0, c]$ of $\square_{Tf}|_{\Omega^i(M)}$, $0 \leq i \leq n$, equals to m_i.*

Proposition 5.5 will be proved in the next section.

We now prove the Morse inequalities by using Proposition 5.5.

For any integer i such that $0 \leq i \leq n$, let

$$\mathrm{F}^{[0,c]}_{Tf,i} \subset \Omega^*(M)$$

denote the m_i dimensional vector space generated by the eigenspaces of $\square_{Tf}|_{\Omega^i(M)}$ associated with eigenvalues in $[0, c]$.

Since

$$d_{Tf}\square_{Tf} = \square_{Tf}d_{Tf} = d_{Tf}d^*_{Tf}d_{Tf}$$

and

$$d^*_{Tf}\square_{Tf} = \square_{Tf}d^*_{Tf} = d^*_{Tf}d_{Tf}d^*_{Tf},$$

one sees that d_{Tf} (resp. d^*_{Tf}) maps each $\mathrm{F}^{[0,c]}_{Tf,i}$ to $\mathrm{F}^{[0,c]}_{Tf,i+1}$ (resp. $\mathrm{F}^{[0,c]}_{Tf,i-1}$). Thus, one has the following finite dimensional subcomplex of $(\Omega^*(M), d_{Tf})$:

$$\left(\mathrm{F}^{[0,c]}_{Tf}, d_{Tf}\right) : 0 \longrightarrow \mathrm{F}^{[0,c]}_{Tf,0} \xrightarrow{d_{Tf}} \mathrm{F}^{[0,c]}_{Tf,1} \xrightarrow{d_{Tf}} \cdots \xrightarrow{d_{Tf}} \mathrm{F}^{[0,c]}_{Tf,n} \longrightarrow 0. \tag{5.15}$$

Moreover, one can prove a Hodge decomposition theorem for this finite dimensional complex (or one can just apply the restriction of the Hodge decomposition theorem for $(\Omega^*(M), d_{Tf})$ to this finite dimensional complex). In particular, for any integer i such that $0 \le i \le n$,

$$\beta_{Tf,i}^{[0,c]} := \dim\left(\frac{\ker d_{Tf}|_{\mathrm{F}_{Tf,i}^{[0,c]}}}{\operatorname{Im} d_{Tf}|_{\mathrm{F}_{Tf,i-1}^{[0,c]}}}\right)$$

equals to $\dim(\ker \square_{Tf}|_{\Omega^i(M)})$, which in turn equals to β_i by (5.10). By Proposition 5.5, this implies the *weak Morse inequalities*.

To prove the strong Morse inequalities, we examine the following decompositions obtained from the complex (5.15): for any integer i such that $0 \le i \le n$,

$$\dim \mathrm{F}_{Tf,i}^{[0,c]} = \dim\left(\ker d_{Tf}|_{\mathrm{F}_{Tf,i}^{[0,c]}}\right) + \dim\left(\operatorname{Im} d_{Tf}|_{\mathrm{F}_{Tf,i}^{[0,c]}}\right)$$

$$= \dim\left(\frac{\ker d_{Tf}|_{\mathrm{F}_{Tf,i}^{[0,c]}}}{\operatorname{Im} d_{Tf}|_{\mathrm{F}_{Tf,i-1}^{[0,c]}}}\right) + \dim\left(\operatorname{Im} d_{Tf}|_{\mathrm{F}_{Tf,i-1}^{[0,c]}}\right) + \left(\operatorname{Im} d_{Tf}|_{\mathrm{F}_{Tf,i}^{[0,c]}}\right). \tag{5.16}$$

From Proposition 5.5 and (5.16), one deduces easily that for any integer i such that $0 \le i \le n$,

$$\sum_{j=0}^{i}(-1)^j m_{i-j} = \sum_{j=0}^{i}(-1)^j \left(\beta_{i-j} + \dim\left(\operatorname{Im} d_{Tf}|_{\mathrm{F}_{Tf,i-j-1}^{[0,c]}}\right)\right.$$

$$\left. + \dim\left(\operatorname{Im} d_{Tf}|_{\mathrm{F}_{Tf,i-j}^{[0,c]}}\right)\right)$$

$$= \sum_{j=0}^{i}(-1)^j \beta_{i-j} + \dim\left(\operatorname{Im} d_{Tf}|_{\mathrm{F}_{Tf,i}^{[0,c]}}\right),$$

from which the *strong Morse inequalities* follows. In particular, we see that when $i = n$, the equality (5.4) holds. □

Clearly, equality (5.4) is a special case of the Poincaré-Hopf index formula proved in Chapter 4.

5.6 Proof of Proposition 5.5

We will proceed as in Sections 4.6 and 4.7, which in turn rely on techniques developed in [BL, Chap. 9], to prove Proposition 5.5.

As in (4.34) and (4.35), in view of Proposition 5.4, for any $T>0$ and critical point $x \in M$ of f, set

$$\alpha_{x,T} = \int_{U_x} \gamma(|\mathbf{y}|)^2 \exp\left(-T|\mathbf{y}|^2\right) dy^1 \wedge \cdots \wedge dy^n,$$

$$\rho_{x,T} = \frac{\gamma(|\mathbf{y}|)}{\sqrt{\alpha_{x,T}}} \exp\left(-\frac{T|\mathbf{y}|^2}{2}\right) dy^1 \wedge \cdots \wedge dy^{n_f(x)}. \tag{5.17}$$

Then $\rho_{x,T} \in \Omega^{n_f(x)}(M)$ is of unit length with compact support contained in U_x.

Let E_T denote the direct sum of the vector spaces generated by $\rho_{x,T}$'s, where x runs through the set of critical points of f. Let $E_T^\perp$ be the orthogonal complement to E_T in $\mathbf{H}^0(M)$. Then $\mathbf{H}^0(M)$ admits an orthogonal splitting

$$\mathbf{H}^0(M) = E_T \oplus E_T^\perp. \tag{5.18}$$

Let p_T, $p_T^\perp$ denote the orthogonal projection operators from $\mathbf{H}^0(M)$ to E_T, $E_T^\perp$ respectively.

As in (4.37), we decompose the Witten deformed operator D_{Tf} by setting

$$D_{T,1} = p_T D_{Tf} p_T, \quad D_{T,2} = p_T D_{Tf} p_T^\perp,$$

$$D_{T,3} = p_T^\perp D_{Tf} p_T, \quad D_{T,4} = p_T^\perp D_{Tf} p_T^\perp. \tag{5.19}$$

As in Section 4.7, the estimates summarized in the following proposition are crucial.

Proposition 5.6 *(i) For any* $T>0$,

$$D_{T,1} = 0; \tag{5.20}$$

(ii) There exists constant $T_1 > 0$, such that for any $s \in E_T^\perp \cap \mathbf{H}^1(M)$, $s' \in E_T$ and $T \geq T_1$, one has

$$\|D_{T,2}s\|_0 \leq \frac{\|s\|_0}{T},$$

$$\|D_{T,3}s'\|_0 \leq \frac{\|s'\|_0}{T}; \tag{5.21}$$

(iii) There exist $T_2 > 0$ and $C > 0$ such that for any $s \in E_T^\perp \cap \mathbf{H}^1(M)$ and $T \geq T_2$,

$$\|D_{Tf}s\|_0 \geq C\sqrt{T}\|s\|_0. \tag{5.22}$$

Proof. (i) Let zero(df) denote the set of critical points of f. Then for any $s \in \mathbf{H}^0(M)$, one verifies directly that

$$p_T s = \sum_{x \in \text{zero}(df)} \langle \rho_{x,T}, s\rangle_{\mathbf{H}^0(M)}\, \rho_{x,T}. \tag{5.23}$$

By (5.17) it is clear that for any $x \in \text{zero}(df)$,

$$D_{Tf}\left(\langle \rho_{x,T}, s\rangle_{\mathbf{H}^0(M)}\rho_{x,T}\right) \in \Omega^{n_f(x)-1}(M) \oplus \Omega^{n_f(x)+1}(M) \tag{5.24}$$

has compact support in U_x. Thus, (5.20) follows.

(ii) This is a special case of Proposition 4.11.

(iii) This is a special case of Proposition 4.12.

The proof of Proposition 5.6 is completed. □

Remark 5.7 Similarly, one can show that the operator $D_{T,1}$ in Section 4.6 is also a zero operator. We did not make this explicit since this fact was not used there.

Now for any positive constant $c > 0$, let $E_T(c)$ denote the direct sum of eigenspaces of D_{Tf} associated with the eigenvalues lying in $[-c, c]$. Clearly, $E_T(c)$ is a finite dimensional subspace of $\mathbf{H}^0(M)$.

Let $P_T(c)$ denote the orthogonal projection operator from $\mathbf{H}^0(M)$ to $E_T(c)$.

Lemma 5.8 *There exist $C_1 > 0$, $T_3 > 0$ such that for any $T \geq T_3$ and any $\sigma \in E_T$,*

$$\|P_T(c)\sigma - \sigma\|_0 \leq \frac{C_1}{T}\|\sigma\|_0. \tag{5.25}$$

Proof. Let $\delta = \{\lambda \in \mathbf{C} : |\lambda| = c\}$ be the counter-clockwise oriented circle. By Proposition 5.6, one deduces that for any $\lambda \in \delta$, $T \geq T_1 + T_2$ and $s \in \mathbf{H}^1(M)$,

$$\|(\lambda - D_{Tf})\, s\|_0 \geq \frac{1}{2}\left\|\lambda p_T s - D_{T,2}p_T^{\perp}s\right\|_0$$

$$+\frac{1}{2}\left\|\lambda p_T^{\perp}s - D_{T,3}p_T s - D_{T,4}p_T^{\perp}s\right\|_0$$

$$\geq \frac{1}{2}\left(\left(c - \frac{1}{T}\right)\|p_T s\|_0 + \left(C\sqrt{T} - c - \frac{1}{T}\right)\left\|p_T^{\perp}s\right\|_0\right). \tag{5.26}$$

By (5.26), one sees that there exist $T_4 > T_1 + T_2$ and $C_2 > 0$ such that for any $T \geq T_4$ and $s \in \mathbf{H}^1(M)$,

$$\|(\lambda - D_{Tf})\, s\|_0 \geq C_2\|s\|_0. \tag{5.27}$$

Thus, for any $T \geq T_4$ and $\lambda \in \delta$,

$$\lambda - D_{Tf} : \mathbf{H}^1(M) \to \mathbf{H}^0(M)$$

is invertible.

Thus, the resolvent $(\lambda - D_{Tf})^{-1}$ is well-defined.

By the basic spectral theorem in operator theory (cf. [D]), one has

$$P_T(c)\sigma - \sigma = \frac{1}{2\pi\sqrt{-1}}\int_{\delta}\left((\lambda - D_{Tf})^{-1} - \lambda^{-1}\right)\sigma d\lambda. \tag{5.28}$$

Now one verifies directly by Proposition 5.6(i) that

$$\left((\lambda - D_{Tf})^{-1} - \lambda^{-1}\right)\sigma = \lambda^{-1}(\lambda - D_{Tf})^{-1} D_{T,3}\sigma. \tag{5.29}$$

From Proposition 5.6(ii) and (5.27), one then deduces that for any $T \geq T_4$ and $\sigma \in E_T$,

$$\left\|(\lambda - D_{Tf})^{-1} D_{T,3}\sigma\right\|_0 \leq C_2^{-1}\|D_{T,3}\sigma\|_0 \leq \frac{1}{C_2 T}\|\sigma\|_0. \tag{5.30}$$

From (5.28)-(5.30), one gets (5.25). □

Remark 5.9 Though there have been used complex numbers in the above proof (by which one needs to complexify the spaces and extend the operators accordingly, though this was not stated explicitly in the proof), one can well stay in the real coefficient category by working with the real part of the right hand side of (5.28). We leave these to the interested reader.

Proof of Proposition 5.5. By applying Lemma 5.8 to the $\rho_{x,T}$'s for $x \in \mathrm{zero}(df)$, one sees easily that when T is large enough, the $P_T(c)\rho_{x,T}$'s for $x \in \mathrm{zero}(df)$ are linearly independent. Thus, there exists $T_5 > 0$ such that when $T \geq T_5$,

$$\dim E_T(c) \geq \dim E_T. \tag{5.31}$$

Now if $\dim E_T(c) > \dim E_T$, then there should exist a nonzero element $s \in E_T(c)$ such that s is perpendicular to $P_T(c)E_T$. That is,

$$\langle s, P_T(c)\rho_{x,T}\rangle_{\mathbf{H}^0(M)} = 0 \tag{5.32}$$

for any $x \in \mathrm{zero}(df)$.

From (5.23) and (5.32), one deduces that

$$\begin{aligned} p_T s = & \sum_{x\in \mathrm{zero}(df)} \langle s, \rho_{x,T}\rangle_{\mathbf{H}^0(M)}\, \rho_{x,T} \\ & - \sum_{x\in \mathrm{zero}(df)} \langle s, P_T(c)\rho_{x,T}\rangle_{\mathbf{H}^0(M)}\, P_T(c)\rho_{x,T} \\ = & \sum_{x\in \mathrm{zero}(df)} \langle s, \rho_{x,T}\rangle_{\mathbf{H}^0(M)}\, (\rho_{x,T} - P_T(c)\rho_{x,T}) \\ & + \sum_{x\in \mathrm{zero}(df)} \langle s, \rho_{x,T} - P_T(c)\rho_{x,T}\rangle_{\mathbf{H}^0(M)}\, P_T(c)\rho_{x,T}. \end{aligned} \tag{5.33}$$

By (5.33) and Lemma 5.8, there exists $C_3 > 0$ such that when $T \geq T_5$,

$$\|p_T s\|_0 \leq \frac{C_3}{T}\|s\|_0. \tag{5.34}$$

Thus, there exists a constant $C_4 > 0$ such that when $T > 0$ is large enough,

$$\left\|p_T^{\perp} s\right\|_0 \geq \|s\|_0 - \|p_T s\|_0 \geq C_4\|s\|_0. \tag{5.35}$$

From (5.35) and Proposition 5.6, one sees that when $T > 0$ is large enough,

$$CC_4\sqrt{T}\|s\|_0 \leq \left\|D_{Tf}p_T^{\perp}s\right\|_0 = \|D_{Tf}s - D_{Tf}p_Ts\|_0$$

$$= \|D_{Tf}s - D_{T,3}s\|_0 \leq \|D_{Tf}s\|_0 + \|D_{T,3}s\|_0$$

$$\leq \|D_{Tf}s\|_0 + \frac{1}{T}\|s\|_0,$$

from which one gets

$$\|D_{Tf}s\|_0 \geq CC_4\sqrt{T}\|s\|_0 - \frac{1}{T}\|s\|_0.$$

Clearly, when $T > 0$ is large enough, this contradicts with the assumption that s is a nonzero element in $E_T(c)$.

Thus, one has

$$\dim E_T(c) = \dim E_T = \sum_{i=0}^{n} m_i. \tag{5.36}$$

Moreover, $E_T(c)$ is generated by $P_T(c)\rho_{x,T}$'s for all $x \in \text{zero}(df)$.

Now in order to prove Proposition 5.5, for any integer i such that $0 \leq i \leq n$, denote by Q_i the orthogonal projection operator from $\mathbf{H}^0(M)$ onto the L^2-completion space of $\Omega^i(M)$. Since $\Box_{Tf}$ preserves the $\mathbf{Z}$-grading of $\Omega^*(M)$, one sees that for any eigenvector s of D_{Tf} associated with an eigenvalue $\mu \in [-c, c]$,

$$\Box_{Tf}Q_is = Q_i\Box_{Tf}s = \mu^2 Q_is.$$

That is, $Q_is \in \Omega^i(M)$ is an eigenvector of $\Box_{Tf}$ associated with eigenvalue μ^2.

Thus, in order to prove Proposition 5.5, one needs only to show that when $T > 0$ is large enough,

$$\dim Q_iE_T(c) = m_i. \tag{5.37}$$

To prove (5.37), one uses Lemma 5.8 to see that for any $x \in \text{zero}(df)$,

$$\left\|Q_{n_f(x)}P_T(c)\rho_{x,T} - \rho_{x,T}\right\|_0 \leq \frac{C_1}{T}. \tag{5.38}$$

From (5.38), one sees that when $T > 0$ is large enough, theforms $Q_{n_f(x)} P_T(c)\rho_{x,T}$, $x \in \mathrm{zero}(df)$, are linearly independent. Thus, for each integer i between 0 and n,

$$\dim Q_i E_T(c) \geq m_i. \tag{5.39}$$

On the other hand, by (5.36) one has

$$\sum_{i=0}^{n} \dim Q_i E_T(c) \leq \dim E_T(c) = \sum_{i=0}^{n} m_i. \tag{5.40}$$

From (5.39) and (5.40), one gets (5.37).

The proof of Proposition 5.5 is completed. □

Remark 5.10 Since the constant $c > 0$ in Proposition 5.5 can be chosen arbitrarily small, one sees that when $T \to +\infty$, the eigenvalues in $[0, c]$ of $\square_{Tf}$ converge to zero.

5.7 Some Remarks and Comments

1). Witten's original paper [W1] was very influential in 1980's. Many rigorous accounts of the analytic proof of the Morse inequalities appeared right after the appearance of [W1]. Here we only mention the paper by Helffer-Sjöstrand [HS] which was based on semi-classical analysis and the paper by Bismut [B] where a proof by heat equation methods was developed. The later also contains an analytic treatment of Bott-Morse inequalities which hold when the critical points are only nondegenerate in the sense of Bott [Bo1].

2). Witten further suggested in [W1] that under some generic conditions, from the complex $(\mathrm{F}_{Tf}^{[0,c]}, d_{Tf})$ defined in (5.15) one can even recover the Thom-Smale complex (cf. [L]) associated to the Morse function f. Witten's this idea, which was proved rigorously in [HS] (Compare also with [BZ2, Sect. 6]), has a tremendous influence on the subsequent developments. For example, it is one of the sources for Floer's conception [F] of Floer homology (cf. [Bo2] for a nice informal account on these). In another direction, Bismut and Zhang [BZ1] used these ideas to give a heat kernel proof, as well as an extension to the case of general flat vector bundles, of the theorems of Cheeger [C] and Müller ([Mü1], [Mü2]) on relations between the Ray-Singer analytic torsion [RS] and the Reidemeister torsion. Most recently, a far reaching generalization of the main results in [BZ1] and [BZ2] to the case of fibrations has been obtained by Bismut and Goette, see [BG1] and [BG2] for more details.

3). In a subsequent paper [W2], Witten also proposed certain holomorphic Morse inequalities for circle actions on Kähler manifolds. These holomorphic Morse inequalities were first proved rigorously by Mathai and Wu [MW] by a heat equation method for the case where the fixed point set of the circle action consists of isolated points. The paper [WZ] contains a proof by using the analytic arguments similar to what in this chapter. It also covers the case where the fixed point set of the circle action may be non-isolated.

4). The analytic localization methods described in Chapters 4 and 5, with necessary technical refinements if needed, are very useful for a wide range of problems in index theory (cf. [BL]). We hope to have shown that the basic ideas involved are in fact very simple.

5.8 References

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